

Universality of the second mixed moment of the characteristic polynomials of the 1D band matrices: real symmetric case

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Abstract

We prove that the asymptotic behavior of the second mixed moment of the characteristic polynomials of the $N \times N$ 1D Gaussian real symmetric band matrices with the width of the band $W \gg N^{1/2}$ coincides with those for the Gaussian Orthogonal Ensemble (GOE). Here we adapt the approach of [17], where the Hermitian case was considered.

1 Introduction

The paper is the continuation of [17] to which we will frequently refer in this paper. In [17] we proved that the asymptotic behavior of the second mixed moment of the characteristic polynomials of the 1D Gaussian Hermitian random band matrices with $W \gg N^{1/2}$ coincides with those for the Hermitian random matrices with i.i.d. (modulo symmetry) Gaussian random entries (GUE). The convenient integral representation for the second correlation function of the characteristic polynomials was obtained there by using the supersymmetry techniques (SUSY). The SUSY method is widely used in the physics literature (see, e.g., [7, 14]) and is potentially very powerful but the rigorous control of the integral representations, which can be obtained by this method, is difficult. So far the most of rigorous results obtained by using the SUSY approach concern the case of unitary symmetry. The goal of this paper is to show that the SUSY approach can be applied to the case of the orthogonal symmetry as well, as to the unitary one.

We consider the real symmetric $N \times N$ matrices H_N (we enumerate indices of entries by $i, j \in \mathcal{L}$, where $\mathcal{L} = [-n, n]^d \cap \mathbb{Z}^d$, $N = (2n + 1)^d$) whose entries H_{ij} are random real Gaussian variables with mean zero such that

$$\mathbb{E}\{H_{ij}H_{lk}\} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})J_{ij}, \quad -n \leq i, j, k, l \leq n, \quad (1.1)$$

where

$$J_{ij} = (-W^2 \Delta + 1)_{ij}^{-1}, \quad (1.2)$$

and Δ is the discrete Laplacian on \mathcal{L} with Neumann boundary conditions (cf. [17], eq. (1.1) – (1.2)). Note that the variance of matrix elements J_{ij} is exponentially small when $|i - j| \gg W$, and so W can be considered as the width of the band. In this paper we will focus on the one-dimensional case ($d = 1$).

The probability law of real symmetric 1D RBM can be written in the form

$$P_N(dH_N) = \prod_{-n \leq i < j \leq n} \frac{dH_{ij}}{\sqrt{2\pi J_{ij}}} e^{-\frac{H_{ij}^2}{2J_{ij}}} \prod_{i=-n}^n \frac{dH_{ii}}{\sqrt{4\pi J_{ii}}} e^{-\frac{H_{ii}^2}{4J_{ii}}}. \quad (1.3)$$

Let $\lambda_1^{(N)}, \dots, \lambda_N^{(N)}$ be the eigenvalues of H_N . Define their Normalized Counting Measure (NCM) as

$$\mathcal{N}_N(\sigma) = \sharp\{\lambda_j^{(N)} \in \sigma, j = 1, \dots, N\}/N, \quad \mathcal{N}_N(\mathbb{R}) = 1, \quad (1.4)$$

where σ is an arbitrary interval of the real axis. The behavior of \mathcal{N}_N as $N \rightarrow \infty$ was studied for many ensembles. For 1D RBM it was shown in [1, 15] that \mathcal{N}_N converges weakly, as $N, W \rightarrow \infty$, to a non-random measure \mathcal{N} , which is called the limiting NCM of the ensemble. The measure \mathcal{N} is absolutely continuous and its density ρ is given by the well-known Wigner semicircle law :

$$\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}, \quad \lambda \in [-2, 2]. \quad (1.5)$$

Random band matrices (RBM) are natural intermediate models to study eigenvalue statistics and quantum propagation in disordered systems as they interpolate between mean-field Wigner matrices (hermitian or real symmetric matrices with i.i.d. random entries) and random Schrödinger operators, where only a random one-site potential is present in addition to the Laplacian on a regular box in \mathbb{Z}^d . In particular, RBM can be used to model the Anderson metal-insulator phase transition.

Let ℓ be the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if ℓ is comparable with the matrix size, and it is called localized otherwise. Delocalized systems correspond to electric conductors, and localized systems are insulators.

According to the physical conjecture (see [4, 11]) for 1D RBM the expected order of ℓ is W^2 (for the energy in the bulk of the spectrum), which means that varying W we can see the crossover: for $W \gg \sqrt{N}$ the eigenvectors are expected to be delocalized and for $W \ll \sqrt{N}$ they are localized. At the present time only some upper and lower bounds for ℓ are proven rigorously. It is known from the paper [16] that $\ell \leq W^8$. On the other side, in the papers [8, 9] it was proven first that $\ell \gg W^{7/6}$, and then that $\ell \gg W^{5/4}$.

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory (see [17] for more details). In this language the conjecture about the crossover for real symmetric 1D RBM states that we get the same local eigenvalue statistics in the bulk of the spectrum as for GOE (real symmetric matrices with i.i.d Gaussian entries) for $W \gg \sqrt{N}$ (which corresponds to delocalized states), and we get another behavior, which is determined by the Poisson statistics, for $W \ll \sqrt{N}$ (and corresponds to localized states). For the general

real symmetric Wigner matrices (i.e., $W = N$) the bulk universality has been proved in [10], [20]. However, in the general case of RBM the question of bulk universality of local spectral statistics is still open even for $d = 1$.

In this paper we consider the correlation functions (or the mixed moments) of characteristic polynomials, which can be defined as

$$F_{2k}(\Lambda) = \int \prod_{s=1}^{2k} \det(\lambda_s - H_N) P_n(dH_N), \quad (1.6)$$

where $P_n(dH_N)$ is defined in (1.3), and $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_{2k}\}$ are real or complex parameters that may depend on N . Although $F_{2k}(\Lambda)$ is not a local object, it is also expected to be universal in some sense. Moreover, correlation functions of characteristic polynomials are expected to exhibit a crossover which is similar to that of local eigenvalue statistic. In particular, for the real symmetric 1D RBM they are expected to have the same local behavior as for GOE for $W \gg \sqrt{N}$, and the different behavior for $W \ll \sqrt{N}$.

As was mentioned before, an additional source of motivation for the current work is the development of the SUSY approach in the context of random operators with non-trivial spatial structures. Although in the case of RBM (and some related types of the Wegner models) the SUSY method has been applied rigorously so far mostly to the density of states (see [5], [6]), the result of [18] for the second correlation function of the block-band matrices gives hope that the method can be applied also for R_k . From the SUSY point of view characteristic polynomials correspond to the so-called fermionic sector of the supersymmetric full model, which describes the correlation functions R_k . So the analysis of the local regime of correlation functions of the characteristic polynomial is an important step towards the proof of the universality of the correlation functions R_k for the case of real symmetric 1D RBM.

The asymptotic local behavior in the bulk of the spectrum of the $2k$ -point mixed moment for GOE is known. It was proved for $k = 1$ by Brézin and Hikami [2], who used the SUSY approach, and for general k by Borodin and Strahov [3], who used different techniques, that

$$F_{2k}(\Lambda_0 + \hat{\xi}/N\rho(\lambda_0)) = C_N \frac{\text{Pf}\{DS(\pi(\xi_i - \xi_j))\}_{i,j=1}^{2k}}{\Delta(\xi_1, \dots, \xi_{2k})} (1 + o(1)),$$

where

$$DS(x) = -\frac{3}{x} \frac{d}{dx} \frac{\sin x}{x} = 3 \left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2} \right), \quad (1.7)$$

$\Delta(\xi_1, \dots, \xi_k)$ is the Vandermonde determinant of ξ_1, \dots, ξ_k , and

$$\hat{\xi} = \text{diag}\{\xi_1, \dots, \xi_{2k}\}, \quad \Lambda_0 = \lambda_0 \cdot I.$$

In particular, for $k = 1$ we have

$$F_2(\Lambda_0 + \hat{\xi}/N\rho(\lambda_0)) = C_N \left(\frac{\sin(\pi(\xi_1 - \xi_2))}{\pi^3(\xi_1 - \xi_2)^3} - \frac{\cos(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2} \right) (1 + o(1)),$$

The last formula was proved also for real symmetric Wigner and general sample covariance matrices (see [12]).

In this paper we obtain the same result for $k = 1$ for matrices (1.1) as $N, W \rightarrow \infty$, $W^2 = N^{1+\theta}$, $0 < \theta \leq 1$ (i.e., $W \gg \sqrt{N}$).

Set

$$\lambda_j = \lambda_0 + \frac{\xi_j}{N\rho(\lambda_0)}, \quad j = 1, 2,$$

where $N = 2n + 1$, $\lambda_0 \in (-2, 2)$, ρ is defined in (1.5), and $\{\xi_1, \xi_2\}$ are real parameters varying in any compact set $K \subset \mathbb{R}$, and define

$$D_2 = \prod_{l=1}^2 F_2^{1/2} \left(\lambda_0 + \frac{\xi_l}{N\rho(\lambda_0)}, \lambda_0 + \frac{\xi_l}{N\rho(\lambda_0)} \right). \quad (1.8)$$

The main result of the paper is the following theorem :

Theorem 1. *Consider the random matrices (1.1) – (1.3) with $W^2 = N^{1+\theta}$, where $0 < \theta \leq 1$. Define the second mixed moment F_2 of the characteristic polynomials as in (1.6). Then we have*

$$\lim_{N \rightarrow \infty} D_2^{-1} F_2 \left(\Lambda_0 + \hat{\xi} / (N\rho(\lambda_0)) \right) = 3 \left(\frac{\sin(\pi(\xi_1 - \xi_2))}{\pi^3(\xi_1 - \xi_2)^3} - \frac{\cos(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2} \right), \quad (1.9)$$

and the limit is uniform in ξ_1, ξ_2 varying in any compact set $K \subset \mathbb{R}$. Here $\rho(\lambda)$ and D_2 are defined in (1.5) and (1.8), $\Lambda_0 = \text{diag} \{ \lambda_0, \lambda_0 \}$, $\lambda_0 \in (-2, 2)$, $\hat{\xi} = \text{diag} \{ \xi_1, \xi_2 \}$.

Theorem 1 is similar to the main Theorem 1 of [17].

The paper is organized as follows. In Sec. 2 we obtain a convenient integral representation for F_2 , using the integration over the Grassmann variables. In Sec. 3 we give the sketch of the proof of Theorem 1. Sec. 4 repeats some auxiliary results of [17] needed for the proof. In Sec. 5 we prove Theorem 1, applying the steepest descent method to the integral representation. Sec. 6 is devoted to the proofs of the auxiliary statements.

1.1 Notation

We denote by C , C_1 , etc. various W and N -independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,

- $N = 2n + 1$;
- $\mathbf{E}\{ \dots \}$ is an expectation with respect to the measure (1.3);
- $U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subset \mathbb{R}$;
- $a_\pm = \pm \frac{\sqrt{4 - \lambda_0^2}}{2} = \pm \pi \rho(\lambda_0)$, $\bar{a}_\pm = (a_\pm, \dots, a_\pm) \in \mathbb{R}^N$, (1.10)

where ρ is defined in (1.5);

- $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; (1.11)

- $\Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \hat{\xi} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad L = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix};$
- $\Lambda_{0,4} = \begin{pmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}, \quad \hat{\xi}_4 = \begin{pmatrix} \hat{\xi} & 0 \\ 0 & \hat{\xi} \end{pmatrix}, \quad L_4 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}; \quad (1.12)$
- $\mathring{U}(2) = U(2)/(U(1) \times U(1)), \quad \mathring{Sp}(2) = Sp(2)/(Sp(1) \times Sp(1))$
- $d\mu$ is the Haar measure on $\mathring{U}(2)$, $d\nu$ is the Haar measure on $\mathring{Sp}(2)$;
- $f(x) = (x + i\lambda_0/2)^2/2 - \log(x - i\lambda_0/2)$ (1.13)
 $f_*(x) = \Re(f(x) - f(a_\pm)) = (x^2 - \lambda_0^2/4 - \log(x^2 + \lambda_0^2/4))/2 - \Re f(a_\pm);$
- Ω_δ is a union of

$$\begin{aligned} \Omega_\delta^+ &= \{\{a_j\}, \{b_j\} : a_j, b_j \in U_\delta(a_+) \ \forall j\}, \\ \Omega_\delta^- &= \{\{a_j\}, \{b_j\} : a_j, b_j \in U_\delta(a_-) \ \forall j\}, \\ \Omega_\delta^\pm &= \{\{a_j\}, \{b_j\} : (a_j \in U_\delta(a_+), b_j \in U_\delta(a_-)) \\ &\quad \text{or } (a_j \in U_\delta(a_-), b_j \in U_\delta(a_+)) \ \forall j\}, \end{aligned} \quad (1.14)$$

where $\delta = W^{-\kappa}$ and $\kappa < \theta/8$.

- $c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2} \cdot \sqrt{1 - \lambda_0^2/4}, \quad c_0 = \Re f(a_+) = \frac{2 - \lambda_0^2}{4}; \quad (1.15)$

- $\mu_\gamma(x) = \exp \left\{ -\frac{1}{2} \sum_{j=-n+1}^n (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=-n}^n x_j^2 \right\}; \quad (1.16)$

- $\langle \dots \rangle_0 = Z_{\delta, \gamma}^{-1} \int_{-\delta W}^{\delta W} (\dots) \cdot \mu_\gamma(x) \prod_{q=-n}^n dx_q, \quad Z_{\delta, \gamma} = \int_{-\delta W}^{\delta W} \mu_\gamma(x) \prod_{q=-n}^n dx_q, \quad (1.17)$

$$\langle \dots \rangle = Z_\gamma^{-1} \int (\dots) \cdot \mu_\gamma(x) \prod_{q=-n}^n dx_q, \quad Z_\gamma = \int \mu_\gamma(x) \prod_{q=-n}^n dx_q,$$

where $\delta > 0$ and $\gamma \in \mathbb{C}$, $\Re \gamma > 0$;

- $\langle \dots \rangle_*$ (and $\langle \dots \rangle_{0,*}$) is (1.17) with $\mu_{\Re \gamma}(x)$ instead of $\mu_\gamma(x)$.

2 Integral representation

In this section we obtain an integral representation for F_2 of (1.6) by using rather standard SUSY techniques, i.e., integrals over the Grassmann variables. Integration over the Grassmann variables has been introduced by Berezin and is widely used in the physics literature. A brief outline of the techniques can be found in [17], Sec. 2.1.

The main result of the section is the following proposition

Proposition 1. *The second correlation function of the characteristic polynomials for 1D real symmetric Gaussian band matrices, defined in (1.6), can be represented as follows:*

$$F_2\left(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)}\right) = -(2\pi^3)^{-N} \det^{-3} J \int \exp\left\{-\frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr}(F_j - F_{j-1})^2\right\} \quad (2.1)$$

$$\times \exp\left\{-\frac{1}{4} \sum_{j=-n}^n \text{Tr}\left(F_j + \frac{i\Lambda_{0,4}}{2} + \frac{i\hat{\xi}_4}{N\rho(\lambda_0)}\right)^2\right\} \prod_{j=-n}^n \det^{1/2}(F_j - i\Lambda_{0,4}/2) \prod_{j=-n}^n dF_j,$$

where $\Lambda_{0,4}$ and $\hat{\xi}_4$ are defined in (1.12), and

$$F_j = \begin{pmatrix} x_j & w_{j1} & 0 & w_{j2} \\ \bar{w}_{j1} & y_j & -w_{j2} & 0 \\ 0 & -\bar{w}_{j2} & x_j & \bar{w}_{j1} \\ \bar{w}_{j2} & 0 & w_{j1} & y_j \end{pmatrix}, \quad dF_j = dx_j dy_j d\Re w_{j1} d\Im w_{j1} d\Re w_{j2} d\Im w_{j2}. \quad (2.2)$$

Moreover, (2.1) can be rewritten in the form

$$F_2\left(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)}\right) = -\frac{C(\xi)\det^{-3} J}{(24\pi)^N} \int \exp\left\{-\frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr}(Q_j^* A_{j,4} Q_j - A_{j-1,4})^2\right\}$$

$$\times \exp\left\{-\sum_{j=-n}^n (f(a_j) + f(b_j)) - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr}(R_j P_{-n})^* A_{j,4} (R_j P_{-n}) \hat{\xi}_4\right\}$$

$$\times \prod_{l=-n}^n (a_l - b_l)^4 d\nu(P_{-n}) d\bar{a} d\bar{b} \prod_{p=-n+1}^n d\nu(Q_p), \quad (2.3)$$

where f is defined in (1.13), $A_{j,4} = \text{diag}\{a_j, b_j, a_j, b_j\}$, $\{R_j\}$ and P_{-n} are 4×4 symplectic matrices, $d\nu(P)$ is the Haar measure on $Sp(2)$, and

$$R_k = \prod_{s=k}^{-n+1} Q_s, \quad C(\xi) = \exp\left\{\frac{\lambda_0(\xi_1 + \xi_2)}{2\rho(\lambda_0)} + \frac{\xi_1^2 + \xi_2^2}{2N\rho(\lambda_0)^2}\right\}. \quad (2.4)$$

Remark 1. Formula (2.1) is valid for any dimension if we change the sum $\sum \text{Tr}(F_j - F_{j-1})^2$ to $\sum \text{Tr}(F_j - F_{j'})^2$, where the last sum runs over all pairs of nearest neighbor j, j' in the volume $\mathcal{L} \subset \mathbb{Z}^d$ (see the definition of RBM (1.1) – (1.2)).

Proof. Representing determinants as integrals over Grassmann variables, we obtain

$$F_2(\Lambda) = \mathbf{E}\left\{\int e^{-\sum_{\alpha=1}^2 \sum_{j,k=-n}^n (\lambda_l - H_n)_{j,k} \bar{\psi}_{j\alpha} \psi_{k\alpha}} \prod_{\alpha=1}^2 \prod_{j=-n}^n d\bar{\psi}_{j\alpha} d\psi_{j\alpha}\right\}$$

$$= \mathbf{E}\left\{\int e^{-\sum_{\alpha=1}^2 \lambda_s \sum_{j=-n}^n \bar{\psi}_{j\alpha} \psi_{j\alpha}} \exp\left\{\sum_{j < k} H_{jk} \sum_{\alpha=1}^2 (\bar{\psi}_{j\alpha} \psi_{k\alpha} + \bar{\psi}_{k\alpha} \psi_{j\alpha})\right.\right.$$

$$\left.\left. + \sum_{j=-n}^n H_{jj} \cdot \sum_{\alpha=1}^2 \bar{\psi}_{j\alpha} \psi_{j\alpha}\right\} \prod_{\alpha=1}^2 \prod_{j=-n}^n d\bar{\psi}_{j\alpha} d\psi_{j\alpha}\right\},$$

where $\{\psi_{j\alpha}\}$, $j = -n, \dots, n$, $\alpha = 1, 2$ are the Grassmann variables ($2n + 1$ variables for each determinant in (1.6)). Here and below we use Greek letters such as α, β etc. for the field index and Latin letters j, k etc. for the position index.

Integrating over the measure (1.3), we get

$$F_2(\Lambda) = \int \prod_{\alpha=1}^2 \prod_{q=-n}^n d\bar{\psi}_{q\alpha} d\psi_{q\alpha} \exp \left\{ - \sum_{\alpha=1}^2 \lambda_{\alpha} \sum_{p=-n}^n \bar{\psi}_{p\alpha} \psi_{p\alpha} \right\} \quad (2.5)$$

$$\times \exp \left\{ \frac{1}{2} \sum_{j < k} J_{jk} (\bar{\psi}_{j1} \psi_{k1} + \bar{\psi}_{j2} \psi_{k2} + \bar{\psi}_{k1} \psi_{j1} + \bar{\psi}_{k2} \psi_{j2})^2 + \sum_{j=-n}^n J_{jj} (\bar{\psi}_{j1} \psi_{j1} + \bar{\psi}_{j2} \psi_{j2})^2 \right\}.$$

Now we will need the Hubbard-Stratonovich transform (see, e.g., [19]). This is a well-known simple trick, which is just the Gaussian integration. In the simplest form it looks as following:

$$e^{a^2/2} = (2\pi)^{-1/2} \int e^{-x^2/2+ax} dx. \quad (2.6)$$

Here a can be complex number or the sum of the products of even numbers of Grassmann variables.

Applying a couple of times (2.6), we can write:

$$\int \exp \left\{ - (J^{-1}x, x)/2 + i \sum_{j=-n}^n x_j \bar{\psi}_{j1} \psi_{j1} \right\} \prod_{j=-n}^n dx_j \quad (2.7)$$

$$= (2\pi)^{N/2} \cdot \det^{1/2} J \cdot \exp \left\{ - \frac{1}{2} \sum_{j,k=-n}^n J_{jk} \bar{\psi}_{j1} \psi_{j1} \bar{\psi}_{k1} \psi_{k1} \right\},$$

$$\int \exp \left\{ - (J^{-1}y, y)/2 + i \sum_{j=-n}^n y_j \bar{\psi}_{j2} \psi_{j2} \right\} \prod_{j=-n}^n dy_j \quad (2.8)$$

$$= (2\pi)^{N/2} \cdot \det^{1/2} J \cdot \exp \left\{ - \frac{1}{2} \sum_{j,k=-n}^n J_{jk} \bar{\psi}_{j2} \psi_{j2} \bar{\psi}_{k2} \psi_{k2} \right\},$$

where $x = \{x_j\}_{j=-n}^n$, $y = \{y_j\}_{j=-n}^n$. In addition,

$$\int \exp \left\{ - (J^{-1}\Re w_1, \Re w_1) - (J^{-1}\Im w_1, \Im w_1) \right\} \quad (2.9)$$

$$\times \exp \left\{ i \sum_{j=-n}^n w_{j1} \bar{\psi}_{j1} \psi_{j2} + i \sum_{j=-n}^n \bar{w}_{j1} \bar{\psi}_{j2} \psi_{j1} \right\} \prod_{q=-n}^n d\Re w_{q1} d\Im w_{q1}$$

$$= \pi^N \cdot \det J \cdot \exp \left\{ - \sum_{j \neq k} J_{jk} \bar{\psi}_{j1} \psi_{j2} \bar{\psi}_{k2} \psi_{k1} - \sum_{j=-n}^n J_{jj} \bar{\psi}_{j1} \psi_{j2} \bar{\psi}_{j2} \psi_{j1} \right\},$$

$$\begin{aligned}
& \int \exp \left\{ - (J^{-1} \Re w_2, \Re w_2) - (J^{-1} \Im w_2, \Im w_2) \right\} \\
& \times \exp \left\{ i \sum_{j=-n}^n w_{j2} \bar{\psi}_{j1} \bar{\psi}_{j2} + i \sum_{j=-n}^n \bar{w}_{j2} \psi_{j1} \psi_{j2} \right\} \prod_{q=-n}^n d\Re w_{q2} d\Im w_{q2} \\
& = \pi^N \cdot \det J \cdot \exp \left\{ - \sum_{j \neq k} J_{jk} \bar{\psi}_{j1} \bar{\psi}_{j2} \psi_{k1} \psi_{k2} - \sum_{j=-n}^n J_{jj} \bar{\psi}_{j1} \bar{\psi}_{j2} \psi_{j1} \psi_{j2} \right\},
\end{aligned} \tag{2.10}$$

where $\Re w_\alpha = \{\Re w_{j\alpha}\}_{j=-n}^n$, $\Im w_\alpha = \{\Im w_{j\alpha}\}_{j=-n}^n$, $\alpha = 1, 2$.

Substituting (2.7) – (2.10) and (1.2) for J_{jk}^{-1} into (2.5), putting $\Lambda = \Lambda_0 + \hat{\xi}/N\rho(\lambda_0)$, and integrating over the Grassmann variables, we obtain

$$\begin{aligned}
F_2 \left(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)} \right) &= -(2\pi^3)^{-N} \det^{-3} J \int \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} (F_j - F_{j-1})^2 \right\} \\
&\times \exp \left\{ - \frac{1}{4} \sum_{j=-n}^n \text{Tr} F_j^2 \right\} \prod_{j=-n}^n \det^{1/2} (F_j - i\Lambda_{0,4} - i\hat{\xi}_4/N\rho(\lambda_0)) \prod_{j=-n}^n dF_j
\end{aligned}$$

with F_j of (2.2) and $\Lambda_{0,4}$, $\hat{\xi}_4$ of (1.12). This gives (2.1) after shifting $F_j \rightarrow F_j + i\Lambda_{0,4}/2 + i\hat{\xi}_4/N\rho(\lambda_0)$. The reason of such a shift is that we need to have saddle-points lying on the contour of the integration (see (1.10)).

The matrices of the form (2.2) have two eigenvalues a_j, b_j of the multiplicity two and can be considered as quaternion 2×2 matrices. In this language F is a quaternion self-dual Hermitian matrix, and it can be diagonalized by the quaternion unitary 2×2 matrices $Sp(2)$ (see, e.g., [13], Chapter 2.4), i.e., unitary 4×4 matrices P which admit the relation

$$P \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} P^t = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Change the variables to $F_j = P_j^* A_{j,4} P_j$, where $P_j \in \mathring{Sp}(2)$ and $A_{j,4} = \text{diag} \{a_j, b_j, a_j, b_j\}$. Then dF_j of (2.2) becomes (see, e.g., [13])

$$\frac{\pi^2}{12} (a_j - b_j)^4 da_j db_j d\nu(P_j),$$

where $d\nu(P_j)$ is the normalized to unity Haar measure on the symplectic group $\mathring{Sp}(2)$.

Thus, we have

$$\begin{aligned}
F_2 \left(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)} \right) &= - \frac{C(\xi) \det^{-3} J}{(24\pi)^N} \int d\bar{a} d\bar{b} \int_{\mathring{Sp}(2)^N} \prod_{j=-n}^n d\nu(P_j) \\
&\times \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} (P_j^* A_{j,4} P_j - P_{j-1}^* A_{j-1,4} P_{j-1})^2 \right\} \\
&\times \exp \left\{ - \frac{1}{4} \sum_{j=-n}^n \text{Tr} \left(A_{j,4} + \frac{i\Lambda_{0,4}}{2} \right)^2 - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} P_j^* A_{j,4} P_j \hat{\xi}_4 \right\} \\
&\times \prod_{k=-n}^n (a_k - i\lambda_0/2) (b_k - i\lambda_0/2) \prod_{k=-n}^n (a_k - b_k)^4,
\end{aligned}$$

where $C(\xi)$ is defined in (2.4), and

$$d\bar{a} = \prod_{j=-n}^n da_j, \quad d\bar{b} = \prod_{j=-n}^n db_j.$$

Now changing the “angle variables” P_j to $Q_j = P_j P_{j-1}^*$, $j = -n+1, \dots, n$ (i.e., the new variables are $P_{-n}, Q_{-n+1}, Q_{-n+2}, \dots, Q_n$), we get (2.3). \square

3 Sketch of the proof of Theorem 1

The strategy of the proof is the same as in [17]. The main difference is that now we perform the integration over $\mathring{Sp}(2)$ instead of $\mathring{U}(2)$, which is much more complicated.

So first we note that the main integrations over a_j, b_j are the same as in [17], eq.(2.11), and so the expected saddle-points for each a_j and b_j are still a_{\pm} (see (1.10)). Moreover, we can use the results of [17], Sec. 4.1 – 4.2, where the properties of the function f and of the complex Gaussian distribution μ_{γ} of (1.16) were studied (see Sec. 4.1).

The second step is to prove that the main contribution to the integral (2.3) is given by the integral Σ over Ω_{δ} (see (1.14)). More precisely, we are going to prove that

$$F_2\left(\Lambda_0 + \frac{\hat{\xi}}{N\rho(\lambda_0)}\right) = -\frac{C(\xi)\det^{-3}J}{(24\pi)^N} \cdot \Sigma \cdot (1 + o(1)), \quad W \rightarrow \infty. \quad (3.1)$$

The bound for the complement $|\Sigma_c|$ can be obtained by inserting the absolute value inside the integral and by performing exactly the integral over the symplectic groups. After this, since we are far from the saddle-points of f , one can control the integral in the same way as in [17] (see Lemma 3).

The next step is the calculation of Σ (see Sec. 5.2, Lemma 4). We are going to show that the main contribution to Σ is given by Σ_{\pm} , i.e., the integral over Ω_{δ}^{\pm} . Consider Ω_{δ}^{\pm} . First note that shifting

$$P_j \rightarrow \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma' \end{pmatrix} P_j$$

for some j (σ' is defined in (1.11)), we can rotate each domain of type

$$\{\{a_j\}, \{b_j\} : (a_j \in U_{\delta}(a_+), b_j \in U_{\delta}(a_-)) \text{ or } (a_j \in U_{\delta}(a_-), b_j \in U_{\delta}(a_+)) \forall j\}$$

to the δ -neighborhood of the point (\bar{a}_+, \bar{a}_-) with \bar{a}_{\pm} of (1.10). Thus, we can consider the contribution over Ω_{δ}^{\pm} as 2^N contributions of the δ -neighborhood of the point (\bar{a}_+, \bar{a}_-) . Consider this neighborhood, and change the variables as

$$\begin{aligned} a_j &\rightarrow a_+ + \tilde{a}_j/W, & |\tilde{a}_j| &\leq \delta W, \\ b_j &\rightarrow a_- + \tilde{b}_j/W, & |\tilde{b}_j| &\leq \delta W, \end{aligned}$$

and set $\tilde{A}_{j,4} = \text{diag} \{\tilde{a}_j, \tilde{b}_j, \tilde{a}_j, \tilde{b}_j\}$. To compute Σ_{\pm} , one has to perform first the integral over the symplectic groups. This integral is some analytic in $\{a_j/W\}, \{\tilde{b}_j/W\}$ function

\mathcal{F} . As in [17], the main idea is to prove that the leading part of this function can be obtained by replacing all Q_s in the “bad” term

$$\exp \left\{ -\frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} \left(\prod_{s=j}^{-n+1} Q_s \cdot P_{-n} \right)^* (L_4 + \tilde{A}_{s,4}/W) \left(\prod_{s=j}^{-n+1} Q_s \cdot P_{-n} \right) \hat{\xi}_4 \right\}$$

with I . To this end, we expand the “bad” term into a series and for each summand, which is analytic in $\{\tilde{a}_j/W\}$, $\{\tilde{b}_j/W\}$, find the bound for its Taylor coefficients (see Lemma 6). This means that we obtain the proper majorant for \mathcal{F} in the sense of [17] (i.e., some function whose Taylor expansion’s coefficients are at least the absolute value of the corresponding coefficient of the Taylor expansion of \mathcal{F}), which helps to change the averaging over the complex measure by the averaging of the majorant over the positive one (see Lemma 2). Then, similarly to [17], we will show that the leading term of Σ_{\pm} is the integral over the Gaussian measures $\mu_{c_{\pm}}$ in $\{a_j\}$ and $\{b_j\}$ variables, and the integral over the symplectic group $d\nu(P_{-n})$ which gives the kernel (1.7). This yields an asymptotic expression for Σ_{\pm} (see Lemma 5).

Also it will be shown in Sec. 5.2.2 that the integrals Σ_+ and Σ_- over Ω_{δ}^+ and Ω_{δ}^- have smaller orders than Σ_{\pm} (see Lemma 8).

4 Preliminary results

In this section we restated the results of [17], Sec. 4.2., where the properties of the complex Gaussian distribution μ_{γ} of (1.16) were studied. All proofs can be found in [17].

First note that the straightforward calculation gives in the small neighborhood of a_{\pm}

$$f(x) - f(a_{\pm}) = c_{\pm}(x - a_{\pm})^2 + s_3(x - a_{\pm})^3 + \dots =: c_{\pm}(x - a_{\pm})^2 + \varphi_{\pm}(x - a_{\pm}), \quad (4.1)$$

where c_{\pm} is defined in (1.15) and $|\varphi_{\pm}(x - a_{\pm})| = O(|x - a_{\pm}|^3)$.

Now set

$$\mu_{\gamma}^{(m)}(x) = \exp \left\{ -\frac{1}{2} \sum_{j=2}^m (x_j - x_{j-1})^2 - \frac{\gamma}{W^2} \sum_{j=1}^m x_j^2 \right\}. \quad (4.2)$$

Lemma 1 ([17], Lemma 3). *We have for any $\gamma \in \mathbb{C}$, $\Re \gamma > 0$*

$$\begin{aligned} 1. \quad Z_{\gamma}^{(m)} &:= \int \mu_{\gamma}^{(m)}(x) \prod_{q=1}^m dx_q = (2\pi)^{m/2} \det^{-1/2}(-\Delta + 2\gamma/W^2) \\ &= (2\pi)^{m/2} \left(\frac{\sqrt{2\gamma}}{W} \sinh \frac{m\sqrt{2\gamma}}{W} \right)^{-1/2} (1 + o(1)). \end{aligned}$$

Moreover, if we set

$$G^{(m)}(\gamma) = \left(-\Delta + \frac{2\gamma}{W^2} \right)^{-1}, \quad (4.3)$$

then

$$|G_{ii}^{(m)}(\gamma)| \leq \frac{C_{\gamma} W}{\sqrt{2\gamma}} \coth \frac{m\sqrt{2\gamma}}{W} (1 + o(1)). \quad (4.4)$$

$$2. \frac{|Z_\gamma^{(m)} - Z_{\delta,\gamma}^{(m)}|}{|Z_\gamma^{(m)}|} = |Z_\gamma^{(m)}|^{-1} \left| \int_{\max |x_i| > \delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^m dx_q \right| \leq C_1 e^{-C_2 \delta^2 W}, \quad W \rightarrow \infty,$$

where $m > CW$, $\delta = W^{-\kappa}$ for sufficiently small $\kappa < \theta/8$, and

$$Z_{\delta,\gamma}^{(m)} = \int_{-\delta W}^{\delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^m dx_q.$$

In addition, for any m

$$|Z_\gamma^{(m)}|^{-1} \left| \int_{|x_k - x_1| > \delta W} \mu_\gamma^{(m)}(x) \prod_{q=1}^m dx_q \right| \leq C_1 e^{-C_2 \delta^2 W}, \quad W \rightarrow \infty,$$

and for $m > CW$ and any $\gamma_1, \gamma_2 \in \mathbb{C}$, $\Re \gamma_1, \Re \gamma_2 > 0$

$$\frac{|Z_{\gamma_1}^{(m)}|}{|Z_{\gamma_2}^{(m)}|} \leq e^{C_1 m/W}, \quad W \rightarrow \infty. \quad (4.5)$$

3. Let $m > C_1 W$, $k \leq Cm/W$, $S = \{i_1, \dots, i_s\} \subset \{1, \dots, m\}$, and $\sum_{l=1}^s k_{i_l} = k$, where $k_l \in \{1, \dots, k\}$. Then

$$|Z_\gamma^{(m)}|^{-1} \left| \int_{\max |x_i| > \delta W} \prod_{j \in S} (x_j/W)^{k_j} \cdot \mu_\gamma^{(m)}(x) \prod_{q=1}^m dx_q \right| \leq e^{-C_1 \delta^2 W}, \quad W \rightarrow \infty,$$

where $\delta = W^{-\kappa}$ for sufficiently small $\kappa < \theta/8$.

Introduce the following partial ordering. Let $\Phi_1(x_1, \dots, x_n)$, $\Phi_2(x_1, \dots, x_n)$ be two analytic functions in some ball centered at 0, and let the coefficients of the Taylor expansion of Φ_2 be non-negative. Then we write

$$\Phi_1 \prec \Phi_2 \quad (4.6)$$

if the absolute value of each coefficient of the Taylor expansion of Φ_1 does not exceed the corresponding coefficient of Φ_2 .

It is easy to see that

$$\Phi_3 \prec \Phi_1, \quad \Phi_4 \prec \Phi_2 \Rightarrow \Phi_3 \Phi_4 \prec \Phi_1 \Phi_2. \quad (4.7)$$

We will need

Lemma 2 ([17], Lemma 8). (i) Let $|\phi_1| \leq CW^{-1}$, $|\phi_2| = o(1)$ and $|\phi_k| \leq C^k$ for some absolute constant $C > 0$. Then

$$\langle \prod_{i=-n}^n (1 + \sum_{l=1}^{\infty} |\phi_l| x_i^l / W^l) \rangle_{0,*} \leq \exp\{C|\phi_2|n/W\}. \quad (4.8)$$

In particular, for $|\phi_1| \leq CW^{-1}$, $|\phi_2| = O(1/W)$ we have

$$\langle \prod_{i=-n}^n (1 + \sum_{l=1}^{\infty} |\phi_l| x_i^l / W^l) \rangle_{0,*} = 1 + o(1).$$

(ii) If

$$\Phi_1(s_1, \dots, s_n) - \Phi_1(0, \dots, 0) \prec \prod_{j=1}^n (1 + q(s_j)) - 1,$$

where $s_i = s(\tilde{a}_i/W, \tilde{a}_{i+1}/W, \dots, \tilde{a}_{i+k}/W, \tilde{b}_i/W, \tilde{b}_{i+1}/W, \dots, \tilde{b}_{i+k}/W)$ is a polynomial with $s(0, \dots, 0) = 0$, k is an n -independent constant, and $q(s) = \sum_{j=1}^{\infty} |c_j| s^j$ with $|c_1| \leq CW^{-1}$, $|c_2| = o(1)$, $|c_l| \leq (C_0)^l$, $l \geq 3$, then

$$|\langle \Phi_1(s_1, \dots, s_n) - \Phi_1(0, \dots, 0) \rangle_0| \leq \langle \prod_{j=1}^n (1 + q(s_j^*)) - 1 \rangle_{0,*} + e^{-Cn/W},$$

where s_j^* is obtained from s_j by replacing the coefficients of s with their absolute values.

4.1 Integration over the symplectic group $\mathring{Sp}(2)$

Proposition 2. (i) Let C be a normal 2×2 matrix with distinct eigenvalues c_1, c_2 and $D = \text{diag}\{d_1, d_2\}$, $d_i \in \mathbb{C}$. Then

$$\int_{U(2)} \exp\{t \text{Tr} CU^* DU\} d\mu(U) = \frac{e^{t(c_1 d_1 + c_2 d_2)} - e^{t(c_1 d_2 + c_2 d_1)}}{t(c_1 - c_2)(d_1 - d_2)}, \quad (4.9)$$

where $t \in \mathbb{C}$ is some constant.

(ii) Let

$$F = \begin{pmatrix} X & w_2 \sigma \\ -\overline{w_2} \sigma & X^t \end{pmatrix}, \quad X = \begin{pmatrix} x & w_1 \\ \overline{w_1} & y \end{pmatrix}, \quad (4.10)$$

$$G = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where σ is defined in (1.11), $x, y \in \mathbb{R}$, $w_1, w_2, d_1, d_2 \in \mathbb{C}$. The matrices of the form (4.10) can be diagonalized by $\mathring{Sp}(2)$ transformation P and have two real eigenvalues a, b of multiplicity two. Moreover, the measure

$$dF = dx dy d\Re w_1 d\Im w_1 d\Re w_2 d\Im w_2,$$

can be represented in the form

$$\frac{\pi^2}{12} (a - b)^4 d\nu(P)$$

with

$$d\nu(P) = 3(1 - 2|V_{12}|^2)^2 d\mu(U) d\mu(V). \quad (4.11)$$

Here $d\mu$ is a Haar measure over $\mathring{U}(2)$,

$$P = \begin{pmatrix} V & 0 \\ 0 & \overline{V} \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi \cdot I & \sin \varphi \cdot e^{i\alpha} \cdot \sigma' \\ -\sin \varphi \cdot e^{-i\alpha} \cdot \sigma' & \cos \varphi \cdot I \end{pmatrix}$$

and

$$U = \begin{pmatrix} \cos \varphi & \sin \varphi \cdot e^{i\alpha} \\ -\sin \varphi \cdot e^{-i\alpha} & \cos \varphi \end{pmatrix}, \quad V = \begin{pmatrix} \cos \phi & \sin \phi \cdot e^{i\beta} \\ -\sin \phi \cdot e^{-i\beta} & \cos \phi \end{pmatrix}.$$

Moreover, if $\tilde{t} = t(c_1 - c_2)(d_1 - d_2)$, then

$$\begin{aligned} \int_{\dot{S}p(2)} \exp\{t \operatorname{Tr} GP^*HP/2\} d\nu(P) \\ = \frac{6}{\tilde{t}^2} \left(e^{t(c_1 d_1 + c_2 d_2)} (1 - 2/\tilde{t}) + e^{t(c_1 d_2 + c_2 d_1)} (1 + 2/\tilde{t}) \right), \end{aligned} \quad (4.12)$$

In addition,

$$\begin{aligned} \int_{\Omega} \exp \left\{ -\frac{t}{4} \operatorname{Tr} (F - G)^2 \right\} \Phi(F) dF \\ = \frac{\pi^2}{t^2} \int_{\hat{\Omega}} \exp \left\{ -\frac{t}{2} \operatorname{Tr} (\hat{Y} - D)^2 \right\} \cdot \Phi(\hat{Y}) \cdot \frac{(y_1 - y_2)^2}{(d_1 - d_2)^2} \\ \times \left(1 - \frac{2}{t(y_1 - y_2)(d_1 - d_2)} \right) dy_1 dy_2, \end{aligned} \quad (4.13)$$

where y_1, y_2 are eigenvalues of F , $\hat{Y} = \operatorname{diag}\{y_1, y_2\}$, and

$$dF = dx dy d\Re w_1 d\Im w_1 d\Re w_2 d\Im w_2.$$

Here $\Phi(F)$ is any function which is invariant over $\dot{S}p(2)$ transformation (i.e., depend only on y_1, y_2), Ω is any $\dot{S}p(2)$ invariant domain such that the eigenvalues of F of the form (4.10) run over the symmetric domain $\hat{\Omega}$.

The proof of this proposition can be found in Sec.6.

5 Proof of the main theorem

In this section we will prove Theorem 1 applying the steepest descent method to the integral representation (2.3).

5.1 The bound for Σ_c

Lemma 3. *Let Σ_c be the part of the integral in (2.3) over the complement of the domain Ω_δ , which is defined in (1.14). Then*

$$|\Sigma_c| \leq C_1 W^{-6N+4} (24\pi)^N e^{-2Nc_0} e^{-C_2 W^{1-2\kappa}},$$

where $\kappa < \theta/8$ and $c_0 = \Re f(a_\pm)$.

Proof. According to (2.3), we have

$$\begin{aligned}
|\Sigma_c| &\leq e^{-2Nc_0} \cdot \int_{\Omega_\delta^C} \exp \left\{ - \sum_{j=-n}^n (f_*(a_j) + f_*(b_j)) \right\} \\
&\quad \times \exp \left\{ - \frac{W^2}{4} \sum_{j=-n+1}^n \text{Tr} (Q_j^* A_{j,4} Q_j - A_{j-1,4})^2 \right\} \\
&\quad \times \prod_{l=-n}^n (a_l - b_l)^4 d\nu(P_{-n}) d\bar{a} d\bar{b} \prod_{p=-n+1}^n d\nu(R_p),
\end{aligned}$$

where f_* and c_0 are defined in (1.13) and (1.15). Here we insert the absolute value inside the integral and use that

$$\left| \exp \left\{ - \frac{i}{2N\rho(\lambda_0)} \sum_{j=-n}^n \text{Tr} (R_j P_{-n})^* A_{j,4} (R_j P_{-n}) \hat{\xi}_4 \right\} \right| = 1.$$

To simplify formulas below, set

$$I_0 = W^{-6N+4} (24\pi)^N e^{-2Nc_0} \cdot \left| \det^{-1} (-\Delta + 2c_+/W^2) \right|. \quad (5.1)$$

As we will see below, I_0 is an order of Σ (see Lemma 4). Also recall that, according to Lemma 1 (1),

$$e^{-C_1 N/W} \leq \left| \det^{-1} (-\Delta + 2c_+/W^2) \right| \leq e^{-C_2 N/W}, \quad (5.2)$$

and that $W^2 = N^{1+\theta}$, $\kappa < \theta/8$, and hence $CN/W \ll W^{1-2\kappa}$.

We are going to prove that

$$|\Sigma_c/I_0| \leq e^{-CW^{1-2\kappa}}. \quad (5.3)$$

Using (4.13), we get (recall that $A_j = \text{diag} \{a_j, b_j, a_j, b_j\}$, $j = -n, \dots, n$ and Ω_δ^C is still a symmetric domain)

$$\begin{aligned}
I_0^{-1} \cdot |\Sigma_c| &\leq \frac{12^{N-1} e^{-2Nc_0}}{W^{4(N-1)} I_0} \int_{\Omega_\delta^C} \exp \left\{ - \frac{W^2}{2} \sum_{j=-n+1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\
&\quad \times \exp \left\{ - \sum_{j=-n}^n (f_*(a_j) + f_*(b_j)) \right\} (a_{-n} - b_{-n})^2 (a_n - b_n)^2 \\
&\quad \times \prod_{j=-n}^n \left(1 - \frac{2}{W^2 (a_j - b_j)(a_{j-1} - b_{j-1})} \right) d\bar{a} d\bar{b}
\end{aligned} \quad (5.4)$$

The first line here is obtained performing recursively the integral over Q_j starting from $j = n$ and going backwards. At each step the integral can be written in the form (4.9),

with a suitable choice of the function f . The last product of (5.4) can be bounded by $\exp\{CN/W^2\}$, thus

$$\begin{aligned}
I_0^{-1} \cdot |\Sigma_c| &\leq \frac{12^{N-1} e^{-2Nc_0} \cdot e^{CN/W^2}}{W^{4(N-1)} I_0} \int_{\Omega_\delta^C} \exp \left\{ -\frac{W^2}{2} \sum_{j=-n+1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\
&\times \exp \left\{ -\sum_{j=-n}^n (f_*(a_j) + f_*(b_j)) \right\} (a_{-n} - b_{-n})^2 (a_n - b_n)^2 d\bar{a} d\bar{b} \quad (5.5) \\
&\leq C \cdot W^4 \cdot (2\pi)^{-N} e^{C_1 N/W} \int_{W\Omega_\delta^C} \exp \left\{ -\frac{1}{2} \sum_{j=-n+1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\
&\times \exp \left\{ -\sum_{j=-n}^n (f_*(a_j/W) + f_*(b_j/W)) \right\} (a_{-n} - b_{-n})^2 (a_n - b_n)^2 d\bar{a} d\bar{b},
\end{aligned}$$

where f_* and c_0 are defined in (1.13) and (1.15). Here in the third line we did the change $a_j \rightarrow a_j/W$, $b_j \rightarrow b_j/W$ and used (5.1) – (5.2).

Now the last integral in (5.5) is the same as in [17], eq. (5.5) and so can be bounded in the same way. □

5.2 Calculation of Σ

Lemma 4. *For the integral Σ over the domain Ω_δ (see (1.14)) we have*

$$\begin{aligned}
\Sigma &= \frac{8\pi^4 \rho(\lambda_0)^4 e^{-2Nc_0} (24\pi)^N}{3W^{6N-4}} \cdot DS(\pi(\xi_1 - \xi_2)) \cdot \left| \det^{-1} \left(-\Delta + \frac{2c_+}{W^2} \right) \right| (1 + o(1)) \quad (5.6) \\
&= 8(\pi\rho(\lambda_0))^4 / 3 \cdot DS(\pi(\xi_1 - \xi_2)) \cdot I_0, \quad W \rightarrow \infty,
\end{aligned}$$

where I_0 is defined in (5.1).

Note that (5.6) together with (5.3) yield

$$|\Sigma_c| \leq e^{-CW^{1-2\kappa}} |\Sigma|,$$

which gives (3.1).

Now using (3.1) and (5.6) we get Theorem 1.

Thus, we are left to compute Σ . We are going to show that the leading term in Σ is given by Σ_\pm , i.e., that the contributions of Σ_+ and Σ_- are smaller.

5.2.1 Calculation of Σ_\pm

Consider the δ -neighborhood of the point (\bar{a}_+, \bar{a}_-) with \bar{a}_\pm of (1.10) and $\delta = W^{-\kappa}$.

Let us show that

Lemma 5. *For the integral Σ_\pm over the domain Ω_δ^\pm of (1.14) we have, as $W \rightarrow \infty$*

$$\Sigma_\pm = \frac{8(\pi\rho(\lambda_0))^4 e^{-2Nc_0} (24\pi)^N}{3W^{6N-4}} \cdot DS(\pi(\xi_1 - \xi_2)) \cdot \left| \det^{-1} \left(-\Delta + \frac{2c_+}{W^2} \right) \right| (1 + o(1)).$$

Proof. Performing the change $a_j - a_+ = \tilde{a}_j/W$, $b_j - a_- = \tilde{b}_j/W$ in (2.3) and using (4.1), we obtain (recall that $a_{\pm} = \pm\pi\rho(\lambda_0)$)

$$\begin{aligned} \Sigma_{\pm} = & \frac{2^N e^{-2Nc_0 - i\pi(\xi_1 - \xi_2)}}{W^{2N}} \int_{|\tilde{a}_j|, |\tilde{b}_j| \leq W^{1-\kappa}} \mu_{c_+}(a) \mu_{c_-}(b) \cdot e^{-\sum_{k=-n}^n (\varphi_+(\tilde{a}_k/W) + \varphi_-(\tilde{b}_k/W))} \\ & \times \int_{\tilde{S}p(2)^N} e^{W^2/2} \sum_{j=-n+1}^n \text{Tr} \left(Q_j^*(L_4 + \tilde{A}_{j,4}/W) Q_j(L_4 + \tilde{A}_{j-1,4}/W) - (L_4 + \tilde{A}_{j,4}/W)(L_4 + \tilde{A}_{j-1,4}/W) \right) \\ & \times \exp \left\{ -\frac{i}{2N\rho(\lambda_0)} \sum_{k=-n}^n \left(\text{Tr} (R_k P_{-n})^* (L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) \hat{\xi}_4 - \text{Tr} L_4 \hat{\xi}_4 \right) \right\} \\ & \times \prod_{l=-n}^n (a_+ - a_- + (\tilde{a}_l - \tilde{b}_l)/W)^4 d\nu(P_{-n}) \prod_{q=-n+1}^n d\nu(Q_q) d\bar{a} d\bar{b}, \end{aligned} \quad (5.7)$$

where $L_4 = \text{diag} \{a_+, a_-, a_+, a_-\}$, $\tilde{A}_{j,4} = \text{diag} \{\tilde{a}_j, \tilde{b}_j, \tilde{a}_j, \tilde{b}_j\}$, and $\mu_{\gamma}(a)$ is defined in (1.16).

Now we are going to integrate over $\{Q_j\}$. Introduce

$$\begin{aligned} F(\bar{a}, \bar{b}, Q) &= -\frac{i}{2\rho(\lambda_0)} \sum_{k=-n}^n \left(\text{Tr} (R_k P_{-n})^* (L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) \hat{\xi}_4 - \text{Tr} L_4 \hat{\xi}_4 \right), \quad (5.8) \\ d\eta_j(Q_j, \tilde{A}_j) &= e^{\frac{W^2}{2} \text{Tr} (Q_j^*(L_4 + \tilde{A}_{j,4}/W) Q_j(L_4 + \tilde{A}_{j-1,4}/W) - (L_4 + \tilde{A}_{j,4}/W)(L_4 + \tilde{A}_{j-1,4}/W))} d\nu(Q_j), \\ d\eta(Q, \tilde{A}) &= \prod_{j=-n+1}^n d\eta_j(Q_j, \tilde{A}_j), \quad I_{\eta}(\tilde{A}) = \int d\eta(Q, \tilde{A}), \\ t_j &= W^2 \left(a_+ - a_- + (\tilde{a}_j - \tilde{b}_j)/W \right) \left(a_+ - a_- + (\tilde{a}_{j-1} - \tilde{b}_{j-1})/W \right), \quad q_j = 6/t_j^2. \end{aligned}$$

According to Proposition 2 we have

$$I_{\eta}(\tilde{A}) = \prod_{j=-n+1}^n q_j \left[1 - \frac{2}{t_j} + e^{-t_j} \left(1 + \frac{2}{t_j} \right) \right]. \quad (5.9)$$

We want to integrate the r.h.s. of (5.7) over $d\eta(Q, \tilde{A})$. To this end, we expand $\exp \{F(\bar{a}, \bar{b}, Q)\}$ into a series with respect to the elements of Q_j , $j = -n+1, \dots, n$. We are going to show that the leading term of the integral is given by the summands without any elements of Q_j .

Lemma 6. *In the notations of (5.8)*

$$\left| \left\langle \left\langle (\exp \{(F(\bar{a}, \bar{b}, Q) - F(0, 0, I))/N\} - 1) \cdot \Pi_1 \cdot \Pi_2 \right\rangle_0 \right\rangle_{\eta} \right| = o(1), \quad N \rightarrow \infty, \quad (5.10)$$

where Π_1, Π_2 are the products of the Taylor's series for $\exp \{\varphi_+(\tilde{a}_j/W)\}$ and for $\exp \{\varphi_-(\tilde{b}_j/W)\}$ and

$$\langle \dots \rangle_{\eta} = \left(\prod_{j=-n+1}^n q_j \right)^{-1} \int (\dots) d\eta(Q, \tilde{A}). \quad (5.11)$$

Proof. Since $\hat{\xi}_4 = \frac{\xi_1 + \xi_2}{2} I_4 + \frac{\xi_1 - \xi_2}{2} L_4$ and $a_+ = -a_- = \pi\rho(\lambda_0)$, we have

$$\begin{aligned} & \text{Tr} (R_k P_{-n})^* (a_+ L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) \hat{\xi}_4 - \text{Tr} (a_+ L_4 + \tilde{A}_{k,4}/W) \hat{\xi}_4 \\ &= \frac{\xi_1 - \xi_2}{2} \text{Tr} ((R_k P_{-n})^* (a_+ L_4 + \tilde{A}_{k,4}/W) (R_k P_{-n}) L_4 - (a_+ L_4 + \tilde{A}_{k,4}/W) L_4) \\ &= 4\pi\rho(\lambda_0)(\xi_2 - \xi_1)(1 + (\tilde{a}_k - \tilde{b}_k)/2\pi\rho(\lambda_0)W) \cdot (|(R_k P_{-n})_{12}|^2 + |(R_k P_{-n})_{14}|^2). \end{aligned}$$

For any 4×4 matrix P introduce

$$S(P) = |P_{12}|^2 + |P_{14}|^2. \quad (5.12)$$

Note that for $P \in Sp(2)$ we have $S(P) \in [0, 1]$.

Rewrite

$$\begin{aligned} & F(\bar{a}, \bar{b}, Q) - F(0, 0, I) \\ &= 2i\pi(\xi_1 - \xi_2) \sum_{k=-n+1}^n (S(R_k P_{-n}) - S(P_{-n})) \cdot \left(1 + \frac{\tilde{a}_k - \tilde{b}_k}{2\pi\rho(\lambda_0)W}\right). \end{aligned} \quad (5.13)$$

Thus, we get

$$\begin{aligned} & \left\langle \exp \left\{ \frac{1}{N} \left(F(\bar{a}, \bar{b}, Q) - F(0, 0, I) \right) \right\} - 1 \right\rangle_\eta \\ &= \sum_{p=1}^{\infty} \frac{C^p}{p! N^p} \sum_{k_1, \dots, k_p} \left\langle \prod_{j=1}^p \left[\left(S(R_{k_j} P_{-n}) - S(P_{-n}) \right) \cdot \left(1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{2\pi\rho(\lambda_0)W} \right) \right] \right\rangle_\eta, \end{aligned}$$

where $\langle \dots \rangle_\eta$ is defined in (5.11). Hence, we have to study

$$\Phi_{k_1, \dots, k_p}(\bar{a}, \bar{b}) = \left\langle \prod_{j=1}^p \left(S(R_{k_j} P_{-n}) - S(P_{-n}) \right) \right\rangle_\eta. \quad (5.14)$$

Let $p < Cn/W$ for some constant C . Introduce i.i.d vectors $\{(x_j, y_j)\}$ such that the density of the distribution has the form

$$\rho(x_j, y_j) = 4(a_+ - a_-)^4 x_j y_j \exp\{-(a_+ - a_-)^2 [x_j^2 + y_j^2]\} \cdot \mathbf{1}_{0 < x_j, y_j < W/2}. \quad (5.15)$$

Introduce matrices

$$\tilde{Q}_j = \mathcal{V}_j \cdot \mathcal{U}_j,$$

where

$$\begin{aligned} \mathcal{V}_j &= \begin{pmatrix} \tilde{V}_j & 0 \\ 0 & \overline{\tilde{V}_j} \end{pmatrix}, \quad \tilde{V}_j = \begin{pmatrix} \tilde{r}_j e^{i\tilde{\sigma}_j} & \tilde{v}_j e^{i\sigma_j} \\ -\tilde{v}_j e^{-i\sigma_j} & \tilde{r}_j e^{-i\tilde{\sigma}_j} \end{pmatrix}, \\ \mathcal{U}_j &= \begin{pmatrix} \tilde{t}_j e^{i\tilde{\theta}_j} I & \tilde{u}_j e^{i\theta_j} \sigma' \\ -\tilde{u}_j e^{-i\theta_j} \sigma' & \tilde{t}_j e^{-i\tilde{\theta}_j} I \end{pmatrix} \end{aligned} \quad (5.16)$$

with

$$\begin{aligned}\tilde{v}_j &= x_j/p_j W, \quad \tilde{u}_j = y_j(1 - 2\tilde{v}_j^2)^{-1/2}/p_j W \\ p_j &= \left(1 + \frac{\tilde{a}_j - \tilde{b}_j}{W(a_+ - a_-)}\right)^{1/2} \left(1 + \frac{\tilde{a}_{j-1} - \tilde{b}_{j-1}}{W(a_+ - a_-)}\right)^{1/2}, \\ \tilde{r}_j &= (1 - \tilde{v}_j^2)^{1/2}, \quad \tilde{t}_j = (1 - \tilde{u}_j^2)^{1/2}\end{aligned}$$

and $\theta_j, \tilde{\theta}_j, \sigma_j, \tilde{\sigma}_j \in [-\pi, \pi)$. Define also

$$d\tilde{\eta}_j = (2\pi)^{-4} \rho(x_j, y_j) dx_j dy_j d\theta_j d\tilde{\theta}_j d\sigma_j d\tilde{\sigma}_j, \quad d\tilde{\eta} = \prod_{j=-n+1}^n d\tilde{\eta}_j. \quad (5.17)$$

Note that

$$\int d\tilde{\eta}_j = (1 - e^{-W^2(a_+ - a_-)^2/4})^2 \leq 1.$$

We need

Lemma 7.

$$\begin{aligned}\tilde{\Phi}_{k_1, \dots, k_p}(\bar{a}, \bar{b}) &:= \left\langle \prod_{j=1}^p \left(S(\tilde{R}_{k_j} \cdot P_{-n}) - S(P_{-n}) \right) \cdot \prod_{i=-n+1}^n (1 - 2|(\tilde{V}_i)_{12}|^2) \right\rangle_{\tilde{\eta}} \\ &= \Phi_{k_1, \dots, k_p}(\bar{a}, \bar{b}) + O(e^{-cW^2}),\end{aligned} \quad (5.18)$$

where $\langle \dots \rangle_{\tilde{\eta}}$ means the integration over $d\tilde{\eta}$ and

$$\tilde{R}_{k_j} = \prod_{l=k_j}^{-n+1} \tilde{Q}_l.$$

The proof of the lemma can be found in Sec. 6.

Denote

$$s_j = 1 - \left(1 + \frac{\tilde{a}_j - \tilde{b}_j}{W(a_+ - a_-)}\right) \left(1 + \frac{\tilde{a}_{j-1} - \tilde{b}_{j-1}}{W(a_+ - a_-)}\right). \quad (5.19)$$

Expanding $\mathcal{V}_j, \mathcal{U}_j$ of (5.16) with respect to s_j we get

$$\mathcal{V}_j = \begin{pmatrix} \tilde{V}_j(0) & 0 \\ 0 & \overline{\tilde{V}_j(0)} \end{pmatrix} + \frac{x_j}{W} \cdot g_v(s_j) \cdot \begin{pmatrix} V_j^1 & 0 \\ 0 & \overline{V_j^1} \end{pmatrix} + \frac{x_j^2}{W^2} \sum_{r=1}^{\infty} \begin{pmatrix} V_j^{(r)} & 0 \\ 0 & \overline{V_j^{(r)}} \end{pmatrix} s_j^r,$$

$$\mathcal{U}_j = \mathcal{U}_j(0) + \frac{y_j}{W} \cdot h_u(s_j) \cdot \begin{pmatrix} 0 & e^{i\theta_j} \sigma' \\ -e^{-i\theta_j} \sigma' & 0 \end{pmatrix} + \frac{y_j^2}{W^2} \sum_{r=1}^{\infty} \begin{pmatrix} U_j^{(r)} & 0 \\ 0 & \overline{U_j^{(r)}} \end{pmatrix} s_j^r,$$

where

$$g_v(s_j) = (1 - s_j)^{-1/2} - 1, \quad h_u(s_j) = (1 - 2x_j^2/W^2)^{-1/2} \left(\left(1 - \frac{s_j}{1 - 2x_j^2/W^2}\right)^{-1/2} - 1 \right). \quad (5.20)$$

Here $\tilde{V}_j(0), \mathcal{U}_j(0)$ are unitary matrices (and hence $\|\tilde{V}_j(0)\| \leq 1, \|\mathcal{U}_j(0)\| \leq 1$),

$$\tilde{V}_j^1 = \begin{pmatrix} 0 & e^{i\sigma_j} \\ -e^{-i\sigma_j} & 0 \end{pmatrix}, \quad \|\tilde{V}_j^{(r)}\| \leq C^r, \quad \|\tilde{U}_j^{(r)}\| \leq C^r \quad (r = 1, 2, \dots),$$

and $\{\tilde{V}_j^{(r)}\}, \{\tilde{U}_j^{(r)}\}$ are diagonal matrices.

Since the integrals of $e^{im\theta_j}$ equal 0 for $m \neq 0$ and 2π for $m = 0$, we conclude that if we replace the coefficients in front of $e^{i\theta_j}$ and $e^{-i\theta_j}$ with the bounds for their absolute values, then, after the averaging with respect to θ_j , the resulting coefficients in front of s_j^k will grow. The same is true for the integral with respect to σ_j . Moreover, for $x_j \in (0, W/2)$

$$g_v(s_j) \prec g_v^1(s_j^*) := \frac{C_1}{1 - C_2 s_j^*},$$

$$h_u(s_j) \prec h_u^1(s_j^*) := \frac{C_3}{1 - C_4 s_j^*}$$

where $C_l, l = 1, \dots, 4$ are n -independent constant and

$$s_j^* = \frac{\tilde{a}_j + \tilde{b}_j + \tilde{a}_{j-1} + \tilde{b}_{j-1}}{W(a_+ - a_-)} + \frac{(\tilde{a}_{j-1} + \tilde{b}_{j-1})(\tilde{a}_j + \tilde{b}_j)}{W^2(a_+ - a_-)^2}.$$

Hence,

$$\begin{aligned} & \tilde{\Phi}_{k_1, \dots, k_p}(\bar{a}, \bar{b}) - \tilde{\Phi}_{k_1, \dots, k_p}(0, 0) \\ & \prec \left\langle \left(\text{Prod}_p(x, \sigma) \text{Prod}_p(y, \theta) - 1 \right) \prod_{j=-n}^n \left(1 - \frac{2x_j^2}{W^2} + \frac{x_j^2}{W^2} \cdot \frac{s_j^*}{1 - s_j^*} \right) \right\rangle_{x_j, y_j, \sigma_j, \theta_j}, \end{aligned}$$

where

$$\begin{aligned} \text{Prod}_p(x, \sigma) &= \prod \left| 1 + \frac{x_j}{W} e^{i\sigma_j} s_j^* g(s_j^*) + \frac{x_j^2}{W^2} s_j^* g(s_j^*) \right|^{2p}, \\ \text{Prod}_p(y, \theta) &= \prod \left| 1 + \frac{y_j}{W} e^{i\theta_j} s_j^* h(s_j^*) + \frac{y_j^2}{W^2} s_j^* h(s_j^*) \right|^{2p} \end{aligned}$$

and $g(t)$ and $h(t)$ are the function of the form $C_1/(1 - C_2 t)$ with positive n -independent C_1, C_2 (we denote the set of such function by $\mathcal{L}[t]$).

In addition,

$$\left\langle \frac{x_j^{2k}}{W^{2k}} \right\rangle_{x_j} \leq \frac{k!}{(a_+ - a_-)^{2k} W^{2k}}, \quad \left\langle \frac{y_j^{2k}}{W^{2k}} \right\rangle_{y_j} \leq \frac{k!}{(a_+ - a_-)^{2k} W^{2k}},$$

and thus we conclude

$$\begin{aligned} & \left\langle \text{Prod}_p(x, \sigma) \cdot \prod_{j=-n}^n \left(1 - \frac{2x_j^2}{W^2} + \frac{x_j^2}{W^2} \cdot \frac{s_j^*}{1 - s_j^*} \right) \right\rangle_{x_j, \sigma_j} \\ & \prec \prod \left(1 + \frac{p}{W^2} s_j^* g_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 g_1(s_j^*)^2 \right), \\ & \left\langle \text{Prod}_p(y, \theta) \right\rangle_{y_j, \theta_j} \prec \prod \left(1 + \frac{p}{W^2} s_j^* h_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 h_1(s_j^*)^2 \right), \end{aligned}$$

where $g_1, h_1 \in \mathcal{L}[t]$. Hence, we obtain

$$\begin{aligned} \tilde{\Phi}_{k_1, \dots, k_p}(\bar{a}, \bar{b}) - \tilde{\Phi}_{k_1, \dots, k_p}(0, 0) &\prec \left[\prod \left(1 + \frac{p}{W^2} s_j^* f_1(s_j^*) + \frac{p^2}{W^2} (s_j^*)^2 f_1(s_j^*)^2 \right) - 1 \right] \\ &+ \left[\prod \left(1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1 - s_j^*} \right) - 1 \right] \end{aligned} \quad (5.21)$$

with some $f_1 \in \mathcal{L}[t]$.

Set

$$\Pi_3 = \prod_{j=1}^p \left(1 + \frac{\tilde{a}_{k_j} - \tilde{b}_{k_j}}{(a_+ - a_-)W} \right), \quad \Pi_{3,*} = \prod_{j=1}^p \left(1 + \frac{\tilde{a}_{k_j} + \tilde{b}_{k_j}}{(a_+ - a_-)W} \right).$$

Then Lemma 2 and (5.21) yield

$$\begin{aligned} &\left| \left\langle (\tilde{\Phi}_{k_1, \dots, k_p}(\bar{a}, \bar{b}) - \tilde{\Phi}_{k_1, \dots, k_p}(0, 0)) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \right\rangle_0 \right| \\ &\leq \left\langle \left(\prod \left(1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} \\ &+ \left\langle \left(\prod \left(1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1 - s_j^*} \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} + e^{-Cn/W} \end{aligned} \quad (5.22)$$

Since $p \leq Cn/W$, we have $2p/W^2 \leq W^{-1}$, $p^2/W^2 = o(1)$. In addition, Π_3 has degree $p < Cn/W$, $|\Pi_3| \leq (1 + \delta)^p$. Hence, we can write

$$\begin{aligned} &\left\langle \left(\prod \left(1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} \\ &\leq (1 + \delta)^p \left\langle \left(\exp \left\{ \sum_{i=-n}^n \left(\frac{Cp}{W^2} \cdot \frac{\tilde{a}_i + \tilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\tilde{a}_i^2 + \tilde{b}_i^2}{W^2} \right) \right\} - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \right\rangle_{0,*} \\ &\leq e^{\delta p} \left\langle \left(\exp \left\{ \sum_{i=-n}^n \left(\frac{Cp}{W^2} \cdot \frac{\tilde{a}_i + \tilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\tilde{a}_i^2 + \tilde{b}_i^2}{W^2} \right) \right\} - 1 \right)^2 \right\rangle_{0,*}^{1/2} \cdot \left\langle \Pi_{1,*}^2 \cdot \Pi_{2,*}^2 \right\rangle_{0,*}^{1/2}, \end{aligned}$$

where Π_1, Π_2 are the products of the Taylor's series for $\exp\{\varphi_+(\tilde{a}_j/W)\}$ and for $\exp\{\varphi_-(\tilde{b}_j/W)\}$, and $\Pi_{1,*}, \Pi_{2,*}$ are obtained from Π_1, Π_2 by changing the coefficients to their absolute values.

The second factor is $1 + o(1)$ (see Lemma 2(i)). Moreover, taking the Gaussian integral of the first factor, we obtain

$$\begin{aligned} &\left\langle \left(\prod \left(1 + \frac{2p}{W^2} s_j f_1(s_j) + \frac{p^2}{W^2} s_j^2 f_1(s_j)^2 \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} \\ &\leq e^{\delta p} \left(\exp \left\{ \frac{cp^2 n}{W^3} \right\} - 1 \right) \leq e^{\delta p} \left(\exp \left\{ \frac{cpn^2}{W^4} \right\} - 1 \right). \end{aligned}$$

Similarly,

$$\left\langle \left(\prod \left(1 + \frac{2}{W^2} + \frac{1}{W^2} \frac{s_j^*}{1 - s_j^*} \right) - 1 \right) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \cdot \Pi_{3,*} \right\rangle_{0,*} \leq e^{\delta p} \left(\exp \left\{ \frac{cn}{W^4} \right\} - 1 \right).$$

Thus, since $p < Cn/W$, in view of (5.22), we get

$$\begin{aligned} & \sum_{p=1}^{Cn/W} \frac{(C_1)^p}{p!N^p} \sum_{k_1, \dots, k_p} \left| \left\langle (\tilde{\Phi}_{k_1, \dots, k_p}(\bar{a}, \bar{b}) - \tilde{\Phi}_{k_1, \dots, k_p}(0, 0)) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \right\rangle_0 \right| \\ & \leq \exp\{C_1 e^{\delta + C_2 n^2/W^4}\} - e^{C_1 e^\delta} + \left(\exp\left\{\frac{cn}{W^4}\right\} - 1 \right) \cdot (e^{C_1 e^\delta} - 1) = o(1). \end{aligned} \quad (5.23)$$

If $p \gg n/W$, then $1/\sqrt{p!} \ll e^{-Cn/W}$, and hence we can replace $\langle \dots \rangle_0$ with $\langle \dots \rangle_{0,*}$ (see Lemma 1) and then take the absolute value under the integral and get the bound

$$e^{C_1 n/W} ([\sqrt{Cn/W}]!)^{-1} \sum_{p=Cn/W}^{\infty} (C_2)^p / \sqrt{p!} = o(1).$$

Let us prove now that

$$\tilde{\Phi}_{k_1, \dots, k_p}(0, 0) = \left\langle \prod_{j=1}^p \left(S(\tilde{R}_{k_j}(0)P_{-n}) - S(P_{-n}) \right) \cdot \prod_{i=-n+1}^n (1 - 2|\tilde{V}_i(0)_{12}|^2) \right\rangle_{\tilde{\eta}} = o(1). \quad (5.24)$$

For the simplicity let us write

$$\tilde{R}_k^0 := \tilde{R}_k(0), \quad \tilde{Q}_{k_1}^0 := \tilde{Q}_{k_1}(0), \quad \tilde{V}_i^0 := \tilde{V}_i(0).$$

Note that $S(P) \in [0, 1]$ for $P \in \mathring{S}p(2)$, and thus

$$\begin{aligned} |1 - 2S(P)| & \leq 1, \\ |S(P_1) - S(P_2)| & \leq 1, \quad P_1, P_2 \in \mathring{S}p(2), \\ |1 - 2|\tilde{V}_i(0)_{12}|^2| & \leq 1. \end{aligned} \quad (5.25)$$

Hence, we have

$$\left| \tilde{\Phi}_{k_1, \dots, k_p}(0, 0) \right| \leq \left\langle \left| S(\tilde{R}_{k_1}^0 P_{-n}) - S(P_{-n}) \right| \right\rangle_{\tilde{\eta}} \leq \left\langle \left(S(\tilde{R}_{k_1}^0 P_{-n}) - S(P_{-n}) \right)^2 \right\rangle_{\tilde{\eta}}^{1/2}. \quad (5.26)$$

In addition,

$$\begin{aligned} & \left\langle \left(S(\tilde{R}_{k_1}^0 P_{-n}) - S(P_{-n}) \right)^2 \right\rangle_{\tilde{\eta}} = \left\langle \left((S(\tilde{R}_{k_1-1}^0 P_{-n}) - S(P_{-n})) \right. \right. \\ & \quad \left. \left. + S(\tilde{Q}_{k_1}^0)(1 - 2S(\tilde{R}_{k_1-1}^0 P_{-n})) + H(\tilde{Q}_{k_1}^0, \tilde{R}_{k_1-1}^0) \right)^2 \right\rangle_{\tilde{\eta}}, \end{aligned} \quad (5.27)$$

where

$$H(P, Q) = \sum_{l \neq s} P_{1l} Q_{l2} \overline{P}_{1s} \overline{Q}_{s2} + \sum_{l \neq s} P_{1l} Q_{l4} \overline{P}_{1s} \overline{Q}_{s4}. \quad (5.28)$$

Since it is easy to check that

$$\left\langle \left((S(\tilde{R}_{k_1-1}^0 P_{-n}) - S(P_{-n})) + S(\tilde{Q}_{k_1}^0)(1 - 2S(\tilde{R}_{k_1-1}^0 P_{-n})) \right) H(\tilde{Q}_{k_1}^0, \tilde{R}_{k_1-1}^0) \right\rangle_{\tilde{\eta}} = \quad (5.29)$$

$$\left\langle H(\tilde{Q}_{k_1}^0, \tilde{R}_{k_1-1}^0)^2 \right\rangle_{\tilde{\eta}} \leq C \langle \tilde{v}_{k_1}(0)^2 \rangle_{\tilde{\eta}_{k_1}} + C \langle \tilde{u}_{k_1}(0)^2 \rangle_{\tilde{\eta}_{k_1}} \leq C_1/W^2, \quad (5.30)$$

$$\left\langle S(\tilde{Q}_{k_1}^0)^2 (1 - 2S(\tilde{R}_{k_1-1}^0 P_{-n}))^2 \right\rangle_{\tilde{\eta}} \leq \left\langle S(\tilde{Q}_{k_1}^0) \right\rangle_{\tilde{\eta}_{k_1}} \leq C/W^2, \quad (5.31)$$

$$\left| \left\langle S(\tilde{Q}_{k_1}^0) \left(S(\tilde{R}_{k_1-1}^0 P_{-n}) - S(P_{-n}) \right) \left(1 - 2S(\tilde{R}_{k_1-1}^0 P_{-n}) \right) \right\rangle_{\tilde{\eta}} \right| \leq \left\langle S(\tilde{Q}_{k_1}^0) \right\rangle_{\tilde{\eta}_{k_1}} \leq C/W^2.$$

This, (5.25) and (5.27) yield

$$\begin{aligned} & \left\langle \left(S(\tilde{R}_{k_1}^0 P_{-n}) - S(P_{-n}) \right)^2 \right\rangle_{\tilde{\eta}} \\ & \leq \left\langle \left(S(\tilde{R}_{k_1-1}^0 P_{-n}) - S(P_{-n}) \right)^2 \right\rangle_{\tilde{\eta}} + C/W^2 \leq \dots \leq CN/W^2 = o(1). \end{aligned} \quad (5.32)$$

Now (5.26) and (5.32) give (5.24).

Therefore,

$$\begin{aligned} & \sum_{p=1}^{Cn/W} \frac{(C_1)^p}{p! N^p} \sum_{k_1, \dots, k_p} \left| \left\langle \tilde{\Phi}_{k_1, \dots, k_p}(0, 0) \cdot \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \right\rangle_0 \right| \\ & \leq \sqrt{\frac{CN}{W^2}} \sum_{p=1}^{Cn/W} \frac{(C_1)^p (1 + \delta)^p}{p!} \leq \sqrt{C_1 N/W^2} = o(1), \end{aligned}$$

which together with (5.23) completes the proof of Lemma 6. \square

Thus, we can change $F(\bar{a}, \bar{b}, Q)$ to $F(0, 0, I)$ in (5.7), and then integrate over η , according to (5.9). We obtain

$$\begin{aligned} \Sigma_{\pm} &= 2^N 6^{N-1} W^{-6N+4} e^{-2Nc_0} \int_{\tilde{S}P(2)} \int_{|\tilde{a}_j|, |\tilde{b}_j| \leq W^{1-\kappa}} \mu_{c_+}(a) \mu_{c_-}(b) \\ & \times \exp \left\{ - \sum_{j=-n}^n \varphi_+(\tilde{a}_j/W) - \sum_{j=-n}^n \varphi_-(\tilde{b}_j/W) \right\} \prod_{j=-n+1}^n \left(1 - \frac{2}{W^2 \Delta_j \Delta_{j-1}} \right) \\ & \times e^{-\frac{i}{2\rho(\lambda_0)} \text{Tr } P_{-n}^* L_4 P_{-n} \hat{\xi}_4} \Delta_{-n}^2 \Delta_n^2 d\nu(P_{-n}) \prod_{q=-n}^n d\tilde{a}_q d\tilde{b}_q (1 + o(1)) \end{aligned} \quad (5.33)$$

Integrating over P_{-n} by the Itsykson-Zuber formula (see Proposition 2) and using Lemma 2, we get finally

$$\begin{aligned} \Sigma_{\pm} &= \frac{2^N 6^{N-1} e^{-2Nc_0} \cdot DS(\pi(\xi_1 - \xi_2))}{W^{6N-4}} \int_{|\tilde{a}_j|, |\tilde{b}_j| \leq W^{1-\kappa}} \prod_{q=-n}^n d\tilde{a}_q d\tilde{b}_q \cdot \mu_{c_+}(a) \mu_{c_-}(b) \\ & \times (a_+ - a_- + (\tilde{a}_{-n} - \tilde{b}_{-n})/W)^2 (a_+ - a_- + (\tilde{a}_n - \tilde{b}_n)/W)^2 (1 + o(1)) \\ & = \frac{8\pi^4 \rho(\lambda_0)^4 e^{-2Nc_0} (24\pi)^N \cdot DS(\pi(\xi_1 - \xi_2))}{3 W^{6N-4}} \left| \det^{-1} \left(-\Delta + \frac{2c_+}{W^2} \right) \right| (1 + o(1)). \end{aligned} \quad (5.34)$$

\square

5.2.2 Σ_+ and Σ_- .

In this section we prove that the integrals Σ_+ and Σ_- over Ω_δ^+ and Ω_δ^- have smaller orders than Σ_\pm .

Lemma 8. *For the integral Σ_+ over the domain Ω_δ^+ of (1.14) we have, as $W \rightarrow \infty$*

$$|\Sigma_+| \leq C W^{-2} |\Sigma_\pm|.$$

The same is valid for the integral Σ_- over the domain Ω_δ^- .

Proof. Consider Ω_δ^+ (Ω_δ^- is similar). Returning to x_j, y_j, w_{j1}, w_{j2} coordinates (see (2.2)), we can write that Ω_δ^+ corresponds to the set

$$\tilde{\Omega}_\delta^+ = \{x_j, y_j, w_{j1}, w_{j2} : x_j, y_j \in U_\delta(a_+), |w_{j1}| \leq \delta, |w_{j2}| \leq \delta\}.$$

Change variables as

$$\begin{aligned} x_j &= a_+ + \frac{\tilde{x}_j}{W}, & w_{j1} &= \frac{\tilde{w}_{j1}}{W}, \\ y_j &= a_+ + \frac{\tilde{y}_j}{W}, & w_{j2} &= \frac{\tilde{w}_{j2}}{W}. \end{aligned}$$

This yields

$$\begin{aligned} \Sigma_+ &= \frac{12^N C(\xi)^{-1}}{\pi^{2N} W^{6N}} \int_{|\tilde{x}_j|, |\tilde{y}_j| \leq W^{1-\kappa}} d\tilde{x} d\tilde{y} \int_{|\tilde{w}_{j1}|, |\tilde{w}_{j2}| \leq W^{1-\kappa}} d\Re \tilde{w}_1 d\Im \tilde{w}_1 d\Re \tilde{w}_2 d\Im \tilde{w}_2 \\ &\times \exp \left\{ - \sum_{j=-n+1}^n \left((\tilde{x}_j - \tilde{x}_{j-1})^2/2 + (\tilde{y}_j - \tilde{y}_{j-1})^2/2 + |\tilde{w}_{j1} - \tilde{w}_{1,j-1}|^2 + |\tilde{w}_{j2} - \tilde{w}_{2,j-1}|^2 \right) \right\} \\ &\times \exp \left\{ - \frac{1}{2} \sum_{j=-n}^n \left(\left(a_+ + \frac{\tilde{x}_j}{W} + \frac{i\lambda_0}{2} + \frac{i\xi_1}{N\rho(\lambda_0)} \right)^2 + \left(a_+ + \frac{\tilde{y}_j}{W} + \frac{i\lambda_0}{2} + \frac{i\xi_2}{N\rho(\lambda_0)} \right)^2 \right) \right\} \\ &\times \exp \left\{ - \sum_{j=-n}^n \left(|\tilde{w}_{j1}|^2/W^2 + |\tilde{w}_{j2}|^2/W^2 \right) \right\} \\ &\times \prod_{j=-n}^n \left(\left(a_+ + \frac{\tilde{x}_j}{W} - \frac{i\lambda_0}{2} \right) \left(a_+ + \frac{\tilde{y}_j}{W} - \frac{i\lambda_0}{2} \right) - \frac{|\tilde{w}_{j1}|^2 + |\tilde{w}_{j2}|^2}{W^2} \right), \end{aligned}$$

which gives after some transformations

$$\begin{aligned} \Sigma_+ &= \frac{12^N e^{-i\pi(\xi_1+\xi_2)}}{\pi^{2N} W^{6N} e^{2Nc_0}} \int_{|\tilde{x}_j|, |\tilde{y}_j| \leq W^{1-\kappa}} d\tilde{x} d\tilde{y} \int_{|\tilde{w}_{j1}|, |\tilde{w}_{j2}| \leq W^{1-\kappa}} d\Re \tilde{w}_1 d\Im \tilde{w}_1 d\Re \tilde{w}_2 d\Im \tilde{w}_2 \\ &\times \mu_{c_+}(\tilde{x}) \cdot \mu_{c_+}(\tilde{y}) \cdot \mu_{c_+}(\sqrt{2}\Re \tilde{w}_1) \cdot \mu_{c_+}(\sqrt{2}\Im \tilde{w}_1) \cdot \mu_{c_+}(\sqrt{2}\Re \tilde{w}_2) \cdot \mu_{c_+}(\sqrt{2}\Im \tilde{w}_2) \\ &\times \exp \left\{ - \sum_{j=-n}^n \left(\frac{i\pi\xi_1}{N} \cdot \frac{\tilde{x}_j}{W} + \phi_+(\tilde{x}_j/W) + \frac{i\pi\xi_2}{N} \cdot \frac{\tilde{y}_j}{W} + \phi_+(\tilde{y}_j/W) \right) \right\} \\ &\times \exp \left\{ \sum_{j=-n}^n \Phi_+(\tilde{x}_j/W, \tilde{y}_j/W, \tilde{w}_{j1}/W, \tilde{w}_{j2}/W) \right\}, \end{aligned}$$

where $\tilde{a}_+ = a_+ - i\lambda_0/2$ and

$$\Phi_+(x, y, w_1, w_2) = \log \left(1 - \frac{|w_1|^2 + |w_2|^2}{(x + \tilde{a}_+)(y + \tilde{a}_+)} \right) + \frac{|w_1|^2 + |w_2|^2}{\tilde{a}_+^2}. \quad (5.35)$$

Set

$$\begin{aligned} d\tilde{\mu}_\gamma = & \mu_{c_+}(\tilde{x}) \mu_\gamma(\tilde{y}) \mu_\gamma(\sqrt{2}\Re\tilde{w}_1) \mu_\gamma(\sqrt{2}\Im\tilde{w}_1) \\ & \times \mu_\gamma(\sqrt{2}\Re\tilde{w}_2) \mu_\gamma(\sqrt{2}\Im\tilde{w}_2) d\Re\tilde{w}_1 d\Im\tilde{w}_1 d\Re\tilde{w}_2 d\Im\tilde{w}_2, \end{aligned}$$

and let $\langle \dots \rangle_{\tilde{\mu}_\gamma}$ and $\langle \dots \rangle_{0, \tilde{\mu}_\gamma}$ be an expectation with respect to $d\tilde{\mu}_\gamma$ over \mathbb{R}^{6N} or over $[-W^{1-\kappa}, W^{1-\kappa}]^{6N}$ respectively. Computing the integral $\int d\tilde{\mu}_{c_+}$ we get

$$\Sigma_+ = \frac{(24\pi)^N e^{-i\pi(\xi_1 + \xi_2)} \det^{-3} D}{W^{6N} e^{2Nc_0}} \left\langle \text{Prod}_1(x) \cdot \text{Prod}_2(y) \cdot \text{Prod}_3 \right\rangle_{0, \tilde{\mu}_{c_+}},$$

where $\text{Prod}_l(x)$ and Prod_3 are the products of Taylor's series of $\exp\{-i\pi\xi_l \tilde{x}_j/(NW) - \phi_+(\tilde{x}_j/W)\}$, $l = 1, 2$ and $\exp\{\Phi_+\}$ respectively, and

$$D = -\Delta + \frac{2c_+}{W^2}.$$

Since according to Lemma 2 we have

$$\left\langle \text{Prod}_1(x) \cdot \text{Prod}_2(y) \right\rangle_{0, \tilde{\mu}_{c_+}} = 1 + o(1),$$

and (see Lemma 1)

$$\det^{-1} D \leq CW,$$

we are left to prove that

$$\left\langle \text{Prod}_1(x) \cdot \text{Prod}_2(y) \cdot (\text{Prod}_3 - 1) \right\rangle_{0, \tilde{\mu}_{c_+}} = o(1). \quad (5.36)$$

Note that the series for $\exp\{\Phi_+\}$ starts from the third order. Therefore, repeating almost literally the proof of Lemma 5 of [17], we can prove that

$$\left\langle \exp \left\{ \sum_{j=-n}^n \Phi_+(\tilde{x}_j/W, \tilde{y}_j/W, \tilde{w}_{j1}/W, \tilde{w}_{j2}/W) \right\} - 1 \right\rangle_{0, \tilde{\mu}_{c_+}} = o(1).$$

The key point of Lemma 5 of [17] was Lemma 6. The only difference in the proof of Lemma 6 of [17] is that now g is a polynomial of all variables together $\tilde{x}_j, \tilde{y}_j, \Re\tilde{w}_{j1}, \Im\tilde{w}_{j1}, \Re\tilde{w}_{j2}, \Im\tilde{w}_{j2}$. But again we can change $\langle \dots \rangle_{0,*}$ to $\langle \dots \rangle_*$, then write

$$\begin{aligned} & \left\langle \exp \left\{ \sum_j g(\tilde{x}_j, \tilde{y}_j, \tilde{w}_{j1}, \tilde{w}_{j2}) \right\} \right\rangle_* - 1 \\ & \leq \sum_{i_1} \left\langle g(\tilde{x}_{i_1}, \tilde{y}_{i_1}, \tilde{w}_{i_11}, \tilde{w}_{i_12}) \cdot \exp \left\{ \sum_j g(\tilde{x}_j, \tilde{y}_j, \tilde{w}_{j1}, \tilde{w}_{j2}) \right\} \right\rangle_*, \end{aligned}$$

and apply the Wick theorem until we get $\langle \exp \left\{ \sum_j g(\tilde{x}_j, \tilde{y}_j, \tilde{w}_{j1}, \tilde{w}_{j2}) \right\} \rangle_*$ or until the number of steps become bigger than s_κ , which is the number such that $W^{-\kappa s_\kappa} \leq W^{-2}$ (see Step 2). Now we are integrating over $d\tilde{\mu}_{\mathbb{R}^c_+}$, i.e., over all variables together, and so each vertex of the multigraph H corresponding to some site j consists of six parts coming from the degree of each variables $\tilde{x}_j, \tilde{y}_j, \Re \tilde{w}_{j1}, \Im \tilde{w}_{j1}, \Re \tilde{w}_{j2}, \Im \tilde{w}_{j2}$. This means that some pairing are forbidden (for example, between vertices corresponding to $(\Re \tilde{w}_{1i_1})^2 \tilde{x}_{i_1}$ and $(\Im \tilde{w}_{2i_2})^2 \tilde{y}_{i_2}$), and some different pairing can correspond to the same multigraph, but since the number of such pairing is finite (since we make the finite number of steps), it does not change the proof (recall that matrix $M_* = -\Delta + \Re \gamma / W^2$ are the same for each set of variables $\{\tilde{x}_j\}, \{\tilde{y}_j\}, \{\Re \tilde{w}_{j1}\}, \{\Im \tilde{w}_{j1}\}, \{\Re \tilde{w}_{j2}\}, \{\Im \tilde{w}_{j2}\}$).

To derive Lemma 5 of [17] from Lemma 6, we should change $|x_j/W|^3$ in the bound of each addition of Σ_k^0 to $|s(w)_j^2 x_j/W^3|$ or $|s(w)_j^2 y_j/W^3|$, where $s(w)_j = \Re w_{j1}, \Im w_{j1}, \Re w_{j2}$ or $\Im w_{j2}$ (note that each summand in the Taylor's series of $\exp\{\Phi_+\}$ has $s(w)_j^2/W^2$ and x_j/W or y_j/W), and use

$$|s(w)^2 x/W^3| \leq \frac{p^{-1}x^2 + ps(w)^4/W^4}{2}$$

instead of

$$|x|^3 \leq \frac{p^{-1}x^2 + px^4}{2}$$

(see eq. (4.23) in [17]).

Then using Lemma 2 we can prove (5.36), thus Lemma 8. \square

This together with Lemma 5 yield Lemma 4.

6 Auxiliary result

Proof of the Proposition 2. Statement (i) is the well-known Harish Chandra/Itsykson-Zuber formula. Its proof can be found, e.g., in [13], Appendix 5.

To prove (4.13) note that one can diagonalize X by unitary transformation and keep Z and T fixed. Indeed, consider any unitary matrix U which diagonalize X . Since $U \in U(2)$, it has the form

$$U = \begin{pmatrix} \cos \varphi \cdot e^{i\theta_1} & \sin \varphi \cdot e^{i\theta_2} \\ -\sin \varphi \cdot e^{i\theta_3} & \cos \varphi \cdot e^{i(\theta_2+\theta_3-\theta_1)} \end{pmatrix}. \quad (6.1)$$

Moreover, we can shift U by any diagonal unitary matrix U_1 . Choose U_1 such that

$$U_0 = UU_1 = \begin{pmatrix} \cos \varphi & \sin \varphi \cdot e^{i\alpha} \\ -\sin \varphi \cdot e^{-i\alpha} & \cos \varphi \end{pmatrix}.$$

Then

$$U_0 \sigma U_0^t = \sigma,$$

and thus

$$\begin{pmatrix} U_0 & 0 \\ 0 & \bar{U}_0 \end{pmatrix} F \begin{pmatrix} U_0 & 0 \\ 0 & \bar{U}_0 \end{pmatrix}^* = \begin{pmatrix} U_0 X U_0^* & w_2 U_0 \sigma U_0^t \\ -\bar{w}_2 \bar{U}_0 \sigma U_0^* & \bar{U}_0 X^t U_0^t \end{pmatrix} = \begin{pmatrix} \hat{X} & w_2 \sigma \\ -\bar{w}_2 \sigma & \hat{X} \end{pmatrix}.$$

Hence, changing $X \rightarrow U_0^* \hat{X} U_0$ (the Jacobian is $\pi/2(x_1 - x_2)^2$) and using (i), we obtain

$$\begin{aligned} I_t(G) &= \frac{\pi}{2} \int_{\Omega_Y} \int_{U(2)} e^{-\frac{t}{2} \text{Tr}(U_0^* \hat{X} U_0 - D)^2 - t|w_2|^2} (x_1 - x_2)^2 \Phi(\hat{X}, w_2) d\hat{X} dw_2 d\bar{w}_2 d\mu(U_0) \\ &= \frac{\pi}{2t} \int_{\Omega_Y} e^{-\frac{t}{2} \text{Tr}(\hat{X} - D)^2 - t|w_2|^2} \cdot \frac{x_1 - x_2}{d_1 - d_2} \cdot (1 - e^{-t(x_1 - x_2)(d_1 - d_2)}) \Phi(\hat{X}, w_2) d\hat{X} dw_2 d\bar{w}_2 \\ &= \frac{\pi}{2t} \int_{\Omega_Y} e^{-\frac{t}{2} \text{Tr}(Y - D)^2} \cdot \frac{\text{Tr} Y L}{d_1 - d_2} \cdot (1 - e^{-t \cdot \text{Tr} Y L \cdot (d_1 - d_2)}) \Phi(Y) dY, \end{aligned}$$

where

$$Y = \begin{pmatrix} x_1 & w_2 \\ \bar{w}_2 & x_2 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad dY = d\hat{X} dw_2 d\bar{w}_2, \quad \Omega_Y = \{Y : F \in \Omega\}.$$

Now diagonalizing Y by the unitary transformation V , writing

$$\text{Tr} V^* \hat{Y} V \mathcal{L} = (y_1 - y_2)(1 - 2|V_{12}|^2)$$

and again using (4.9), we get finally

$$\begin{aligned} I_t(G) &= \frac{\pi^2}{4t} \int_{\hat{\Omega}} \int_{U(2)} \exp \left\{ -\frac{t}{2} \text{Tr} (V^* \hat{Y} V - D)^2 \right\} \cdot \frac{1 - 2|V_{12}|^2}{d_1 - d_2} \\ &\times \left(1 - \exp \left\{ -t \text{Tr} V^* \hat{Y} V \mathcal{L} \cdot (d_1 - d_2) \right\} \right) (y_1 - y_2)^3 dy_1 dy_2 d\mu(V) \\ &= \frac{\pi^2}{4t^2} \int dy_1 dy_2 \exp \left\{ -\frac{t}{2} \text{Tr} (\hat{Y}^2 + D^2) \right\} \cdot \Phi(\hat{Y}) \cdot \frac{(y_1 - y_2)^2}{(d_1 - d_2)^2} \\ &\times \left[e^{t(y_1 d_1 + y_2 d_2)} \cdot \left(1 - \frac{2}{t(y_1 - y_2)(d_1 - d_2)} \right) + e^{t(y_1 d_2 + y_2 d_1)} \cdot \left(1 + \frac{2}{t(y_1 - y_2)(d_1 - d_2)} \right) \right], \end{aligned}$$

which, taking into account the symmetry of $\hat{\Omega}$, yields (4.13). Integral (4.12) can be computed straightforward. \square

Proof of Lemma 7. Note that all non-zero moments of measure $d\eta$ can be expressed via expectations of $v_j^{2s} := |(V_j)_{12}|^{2s}$, $u_j^{2l} := |(U_j)_{12}|^{2l}$. In addition, according to Proposition 2,

$$\begin{aligned} \langle v_j^{2s} u_j^{2l} \rangle_{\eta_j} &= 12q_j^{-1} \int_0^1 v_j^{2s+1} u_j^{2l+1} e^{t_j(1-2v_j^2)(1-2u_j^2)/2-t_j/2} (1-2v_j^2)^2 du_j dv_j \\ &= 24q_j^{-1} \int_0^1 du_j \int_0^{1/\sqrt{2}} dv_j v_j^{2s+1} u_j^{2l+1} e^{t_j(1-2v_j^2)(1-2u_j^2)/2-t_j/2} (1-2v_j^2)^2 \\ &= \frac{24q_j^{-1}}{W^4 p_j^4} \int_0^{p_j W/\sqrt{2}} v_j dv_j \int_0^{p_j W \sqrt{1-2v_j^2/p_j^2}} u_j du_j \left(\frac{v_j}{W p_j} \right)^{2s} \cdot \left(\frac{u_j}{W p_j (1-2v_j^2/W^2 p_j^2)^{1/2}} \right)^{2l} \\ &\times \exp \left\{ -(a_+ - a_-)^2 (v_j^2 + u_j^2) \right\} \cdot \left(1 - \frac{2v_j^2}{W^2 p_j^2} \right) \\ &= \langle \tilde{v}_j^{2s} \tilde{u}_j^{2l} \cdot (1 - 2\tilde{v}_j^2) \rangle_{\tilde{\eta}_j} + O(e^{-C_1 W^2}), \end{aligned}$$

where η_j , q_j and t_j are defined in (5.8), and in the third line we have changed $t_j v_j^2 \rightarrow (a_+ - a_-)^2 v_j^2$, $t_j(1 - 2v_j^2)u_j^2 \rightarrow (a_+ - a_-)^2 u_j^2$.

Now let \mathbf{E}_k be the averaging with respect to the product of the measures $d\tilde{\eta}_j$ for j from $(-n+1)$ to $(-n+k)$ and the measures $d\eta_j$ for j from $(-n+k+1)$ to n . Thus, if

$$\Psi_{k_1, \dots, k_s} = \prod_{j=1}^s S(R_{k_j} P_{-n}),$$

then it suffices to estimate

$$\left| \tilde{\Psi}_{k_1, \dots, k_s}^0 - \tilde{\Psi}_{k_1, \dots, k_s}^{2n} \right| \leq e^{-cW^2}$$

for $s \leq p$, where

$$\tilde{\Psi}_{k_1, \dots, k_s}^i = \mathbf{E}_i \left\{ \Psi_{k_1, \dots, k_s} \prod_{j=-n+1}^{-n+i} (1 - 2\tilde{v}_j^2) \right\}.$$

Note that

$$\left| \tilde{\Psi}_{k_1, \dots, k_s}^0 - \tilde{\Psi}_{k_1, \dots, k_s}^{2n} \right| \leq \sum_{i=1}^{2n} \left| \tilde{\Psi}_{k_1, \dots, k_s}^{i-1} - \tilde{\Psi}_{k_1, \dots, k_s}^i \right|.$$

In each summand we write for $\gamma = i-1, i$ (we assume that all $k_j \geq (-n+i)$)

$$\begin{aligned} \tilde{\Psi}_{k_1, \dots, k_s}^\gamma &= \mathbf{E}_\gamma \left\{ \prod_{j=1}^s S(R_{-n+i-1} Q_{-n+i} (R_{-n+i}^* R_{k_j} P_{-n})) \prod_{j=-n+1}^{-n+\gamma} (1 - 2\tilde{v}_j^2) \right\} \\ &= \mathbf{E}_\gamma \left\{ \prod_{j=1}^s \sum_{l=2,4} \left| \sum_{\alpha, \alpha'=1, \dots, 4} (R_{-n+i-1})_{1\alpha} (Q_{-n+i})_{\alpha\alpha'} (R_{-n+i}^* R_{k_j} P_{-n})_{\alpha'l} \right|^2 \prod_{j=-n+1}^{-n+\gamma} (1 - 2\tilde{v}_j^2) \right\} \\ &= \begin{cases} \sum_{k,l=1}^{s+1} C_{k,l} \mathbf{E}_\gamma \{ |(V_{-n+i})_{12}|^{2k} |(U_{-n+i})_{12}|^{2l} \}, & \gamma = i-1, \\ \sum_{k,l=1}^{s+1} C_{k,l} \mathbf{E}_\gamma \{ |(V_{-n+i})_{12}|^{2k} |(U_{-n+i})_{12}|^{2l} (1 - 2\tilde{v}_{-n+i}^2) \}, & \gamma = i, \end{cases} \end{aligned}$$

where the coefficients $C_{k,l}$ are the same for $\gamma = i$ and $\gamma = i-1$ and can be bounded by C^s , since $|(R_{-n+i-1})_{1\alpha}| \leq 1$ and $|(R_{-n+i}^* R_{k_j} P_{-n})_{\alpha'l}| \leq 1$, $l = 2, 4$. Moreover, since

$$\begin{aligned} &|\mathbf{E}_{i-1} \{ |(V_{-n+i})_{12}|^{2k} |(U_{-n+i})_{12}|^{2l} \} \\ &\quad - \mathbf{E}_i \{ |(V_{-n+i})_{12}|^{2k} |(U_{-n+i})_{12}|^{2l} (1 - 2\tilde{v}_{-n+i}^2) \}| \leq C^s k! l! e^{-cW^2}, \end{aligned}$$

we obtain

$$\left| \tilde{\Psi}_{k_1, \dots, k_s}^0 - \tilde{\Psi}_{k_1, \dots, k_s}^{2n} \right| \leq n C_1^p (p!)^2 e^{-cW^2} \leq n e^{C_2(n \log n)/W} e^{-cW^2} = O(e^{-C_2 W^2}).$$

This yields Lemma 7. \square

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